

CHARACTERIZATIONS OF ABEL-GRASSMANN'S GROUPOIDS BY THEIR INTUITIONISTIC FUZZY IDEALS

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Abstract. In this paper, we have introduced the concept of intuitionistic fuzzy ideals in an AG-groupoids. We have characterized regular and intra-regular AG-groupoids in terms of intuitionistic fuzzy left (right, two-sided) ideals, fuzzy (generalized) bi-ideals and intuitionistic fuzzy $(1, 2)$ -ideals. We have proved that all the intuitionistic fuzzy ideals coincides in regular and intra-regular AG-groupoids. It has been shown that the set of intuitionistic fuzzy two sided ideals of a regular AG-groupoid forms a semilattice structure. We have also given some useful conditions for an AG-groupoid to become an intra-regular AG-groupoid in terms their intuitionistic fuzzy ideals.

Keywords. AG-groupoids, regular AG-groupoids, intra-regular AG-groupoids and intuitionistic fuzzy ideals.

Introduction

The concept of fuzzy sets was first proposed by Zadeh [19] in 1965. Several researchers were conducted on the generalization of the notion of fuzzy set. Given a set S , a fuzzy subset of S is, by definition an arbitrary mapping $f : S \rightarrow [0, 1]$ where $[0, 1]$ is the unit interval. A fuzzy subset is a class of objects with a grades of membership. The concept of fuzzy set was applied in [2] to generalize the basic concepts of general topology. A. Rosenfeld [16] was the first who consider the case of a groupoid in terms of fuzzy sets. Kuroki has been first studied the fuzzy sets on semigroups [12].

As an important generalization of the notion of fuzzy set, Atanassov [1], introduced the concept of an intuitionistic fuzzy set. De et al. [4] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Dengfeng and Chunfian [5] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similarity measure and discussed applications of the similarity measures between intuitionistic fuzzy sets to pattern recognition problems. Intuitionistic fuzzy sets have many applications in mathematics, Davvaz et al. [3], applied this concept in H_v -modules. They introduced the notion of an intuitionistic fuzzy H_v -submodule of an H_v -module and

studied the related properties. Jun in [7], introduced the concept of an intuitionistic fuzzy bi-ideal in ordered semigroups and characterized the basic properties of ordered semigroups in terms of intuitionistic fuzzy bi-ideals. In [10] and [11], Kim and Jun introduced the concept of intuitionistic fuzzy interior ideals of semigroups. In [17], Shabir and Khan gave the concept of an intuitionistic fuzzy interior ideal of ordered semigroups and characterized different classes of ordered semigroups in terms of intuitionistic fuzzy interior ideals. They also gave the concept of an intuitionistic fuzzy generalized bi-ideal in [18] and discussed different classes of ordered semigroups in terms of intuitionistic fuzzy generalized bi-ideals.

In this paper, we consider the intuitionistic fuzzification of the concept of several ideals in AG-groupoid and investigate some properties of such ideals.

An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. The left identity in an AG-groupoid if exists is unique [14]. An AG-groupoid is non-associative and non-commutative algebraic structure, nevertheless, it posses many interesting properties which we usually find in associative and commutative algebraic structures. An AG-groupoid with right identity becomes a commutative monoid [14]. An AG-groupoid is basically the generalization of semigroup (see [8]) with wide range of applications in theory of flocks [15]. The theory of flocks tries to describes the human behavior and interaction.

The concept of an Abel-Grassmann's groupoid (AG-groupoid) [8] was first given by M. A. Kazim and M. Naseeruddin in 1972 and they called it left almost semigroup (LA-semigroup). P. Holgate call it simple invertive groupoid [6]. An AG-groupoid is a groupoid having the left invertive law,

$$(1) \quad (ab)c = (cb)a, \text{ for all } a, b, c \in S.$$

In an AG-groupoid, the medial law [8] holds,

$$(2) \quad (ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in S.$$

In an AG-groupoid S with left identity, the paramedial law holds,

$$(3) \quad (ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in S.$$

If an AG-groupoid contains a left identity, the following law holds,

$$(4) \quad a(bc) = b(ac), \text{ for all } a, b, c \in S.$$

Preliminaries

Let S be an AG-groupoid, by an AG-subgroupoid of S , we means a non-empty subset A of S such that $A^2 \subseteq A$.

A non-empty subset A of an AG-groupoid S is called left (right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$).

A non-empty subset A of an AG-groupoid S is called two-sided ideal or simply ideal if it is both a left and a right ideal of S .

A non-empty subset A of an AG-groupoid S is called generalized bi-ideal of S if $(AS)A \subseteq A$.

An AG-subgroupoid A of S is called bi-ideal of S if $(AS)A \subseteq A$.

An AG-subgroupoid A of an AG-groupoid S is called $(1, 2)$ ideal of S if $(AS)A^2 \subseteq A$.

Let S be a non empty set, a fuzzy subset f of S is, by definition an arbitrary mapping $f : S \rightarrow [0, 1]$ where $[0, 1]$ is the unit interval. A fuzzy subset f is a class of objects with a grades of membership having the form

$$f = \{(x, f(x))/x \in S\}.$$

An intuitionistic fuzzy set (briefly, *IFS*) A in a non empty set S is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x))/x \in S\}.$$

The functions $\mu_A : S \rightarrow [0, 1]$ and $\gamma_A : S \rightarrow [0, 1]$ denote the degree of membership and the degree of nonmembership respectively such that for all $x \in S$, we have

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1.$$

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the *IFS* $A = \{(x, \mu_A(x), \gamma_A(x))/x \in S\}$.

Let $\delta = \{(x, S_\delta(x), \Theta_\delta(x))/S_\delta(x) = 1 \text{ and } \Theta_\delta(x) = 0 \text{ for all } x \in S\} = (S_\delta, \Theta_\delta)$ be an *IFS*, then $\delta = (S_\delta, \Theta_\delta)$ will be carried out in operations with an *IFS* $A = (\mu_A, \gamma_A)$ such that S_δ and Θ_δ will be used in collaboration with μ_A and γ_A respectively.

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy AG-subgroupoid of S if $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in S$.

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy left ideal of S if $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$ for all $x, y \in S$.

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy right ideal of S if $\mu_A(xy) \geq \mu_A(x)$ and $\gamma_A(xy) \leq \gamma_A(x)$ for all $x, y \in S$.

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called fuzzy two-sided ideal of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S .

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy generalized bi-ideal of S if $\mu_A((xa)y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A((xa)y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all x, a and $y \in S$.

An intuitionistic fuzzy AG-subgroupoid $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy bi-ideal of S if $\mu_A((xa)y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A((xa)y) \leq \gamma_A(x) \vee \gamma_A(y)$ for all x, a and $y \in S$.

An intuitionistic fuzzy AG-subgroupoid $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy $(1, 2)$ -ideal of S if $\mu_A((xw)(yz)) \geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_A(z)$ and $\gamma_A((xw)(yz)) \leq \gamma_A(x) \vee \gamma_A(y) \vee \gamma_A(z)$ for all x, a and $y \in S$.

Let S be an AG-groupoid and let $\phi \neq A \subseteq S$, then the intuitionistic characteristic function $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ of A is defined as

$$\mu_{\chi_A}(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \quad \text{and} \quad \gamma_{\chi_A}(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \notin A \end{cases}$$

It is clear that γ_{χ_A} acts as a complement of μ_{χ_A} , that is, $\gamma_{\chi_A} = \mu_{\chi_{A^c}}$.

Note that in an AG-groupoid S with left identity, $S = S^2$.

An element a of an AG-groupoid S is called regular if there exists $x \in S$ such that $a = (ax)a$ and S is called regular if every element of S is regular.

Example 1. Let $S = \{a, b, c, d, e\}$ be an AG-groupoid with left identity d with the following multiplication table.

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	d	e	c
d	a	b	c	d	e
e	a	b	e	c	d

By routine calculation, it is easy to check that S is regular.

Define an *IFS* $A = (\mu_A, \gamma_A)$ of S as follows: $\mu_A(a) = 1$, $\mu_A(b) = \mu_A(c) = \mu_A(d) = \mu_A(e) = 0$, $\gamma_A(a) = 0.3$, $\gamma_A(b) = 0.4$ and $\gamma_A(c) = \gamma_A(d) = \gamma_A(e) = 0.2$, then clearly $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S .

Theorem 1. *Let S be an AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ be any IFS of S , then S is regular if $A(x) = A(x^2)$ holds for all x in S .*

Proof. Assume that S be an AG-groupoid with left identity. Clearly x^2S is a subset of S and therefore its characteristic function $\chi_{x^2S} = (\mu_{\chi_{x^2S}}, \gamma_{\chi_{x^2S}})$ is an *IFS* of S . Let $x \in S$, then by given assumption $\mu_{\chi_{x^2S}}(x) = \mu_{\chi_{x^2S}}(x^2)$ and $\gamma_{\chi_{x^2S}}(x) = \gamma_{\chi_{x^2S}}(x^2)$ holds for all $x \in S$. As $x^2 \in x^2S$, because by using (3), we have

$$x^2S = (xx)(SS) = (SS)(xx) = Sx^2.$$

Therefore $\mu_{\chi_{x^2S}}(x^2) = 1$ and $\gamma_{\chi_{x^2S}}(x^2) = 0$, which implies that $x \in x^2S$. Now by using (1), (4) and (3), we have

$$\begin{aligned} x &\in x^2S = (xx)(SS) = ((SS)x)x \subseteq ((SS)(x^2S))x = ((SS)((xx)S))x \\ &= ((SS)((Sx)x))x = ((Sx)(Sx))x = ((xS)(xS))x = (x((xS)S))x \subseteq (xS)x. \end{aligned}$$

Thus S is regular. \square

The converse is not true in general. For this let us consider a regular AG-groupoid S in Example 1. Define an *IFS* $A = (\mu_A, \gamma_A)$ of S as follows: $\mu_A(a) = 0.6$, $\mu_A(b) = 0.2$, $\mu_A(c) = \mu_A(d) = \mu_A(e) = 0.9$, $\gamma_A(a) = 0.7$, $\gamma_A(b) = 0.3$ and $\gamma_A(c) = \gamma_A(d) = \gamma_A(e) = 1$, then it is easy to see that $\mu_A(a) \neq \mu_A(a^2)$ and $\gamma_A(a) \neq \gamma_A(a^2)$, that is, $A(a) \neq A(a^2)$ for $a \in S$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are *IFSs* of an AG-groupoid S . The symbols $A \cap B$ will means the following *IFS* of S

$$\begin{aligned} (\mu_A \cap \mu_B)(x) &= \min\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x), \text{ for all } x \text{ in } S. \\ (\gamma_A \cup \gamma_B)(x) &= \max\{\gamma_A(x), \gamma_B(x)\} = \gamma_A(x) \vee \gamma_B(x), \text{ for all } x \text{ in } S. \end{aligned}$$

The symbols $A \cup B$ will means the following *IFS* of S

$$\begin{aligned} (\mu_A \cup \mu_B)(x) &= \max\{\mu_A(x), \mu_B(x)\} = \mu_A(x) \vee \mu_B(x), \text{ for all } x \text{ in } S. \\ (\gamma_A \cap \gamma_B)(x) &= \min\{\gamma_A(x), \gamma_B(x)\} = \gamma_A(x) \wedge \gamma_B(x), \text{ for all } x \text{ in } S. \end{aligned}$$

$A \subseteq B$ means that

$$\mu_A(x) \leq \mu_B(x) \text{ and } \gamma_A(x) \geq \gamma_B(x) \text{ for all } x \text{ in } S.$$

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any two *IFSs* of an AG-groupoid S , then the product $A \circ B$ is defined by,

$$(\mu_A \circ \mu_B)(a) = \begin{cases} \bigvee_{a=bc} \{\mu_A(b) \wedge \mu_B(c)\}, & \text{if } a = bc \text{ for some } b, c \in S. \\ 0, & \text{otherwise.} \end{cases}$$

$$(\gamma_A \circ \gamma_B)(a) = \begin{cases} \bigwedge_{a=bc} \{\gamma_A(b) \vee \gamma_B(c)\}, & \text{if } a = bc \text{ for some } b, c \in S. \\ 1, & \text{otherwise.} \end{cases}$$

Lemma 1. ([13],[9]) *Let S be an AG-groupoid, then the following holds.*

(i) An *IFS* $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy AG-subgroupoid of S if and only if $\mu_A \circ \mu_A \subseteq \mu_A$ and $\gamma_A \circ \gamma_A \supseteq \gamma_A$.

(ii) An *IFS* $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy left (right) ideal of S if and only if $S \circ \mu_A \subseteq \mu_A$ and $\Theta \circ \gamma_A \supseteq \gamma_A$ ($\mu_A \circ S \subseteq \mu_A$ and $\gamma_A \circ \Theta \supseteq \gamma_A$).

Lemma 2. *Let S be an AG-groupoid and let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are any intuitionistic fuzzy two sided ideals of S , then $A \circ B = A \cap B$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are any intuitionistic fuzzy two sided ideals of a regular AG-groupoid S , then by using Lemma 1, we have $\mu_A \circ \mu_B \subseteq \mu_A \cap \mu_B$ and $\gamma_A \circ \gamma_B \supseteq \gamma_A \cup \gamma_B$, which shows that $A \circ B \subseteq A \cap B$. Let $a \in S$, then there exists $x \in S$ such that $a = (ax)a$ and therefore, we have

$$\begin{aligned} (\mu_A \circ \mu_B)(a) &= \bigvee_{a=(ax)a} \{\mu_A(ax) \wedge \mu_B(a)\} \geq \mu_A(ax) \wedge \mu_B(a) \\ &\geq \mu_A(a) \wedge \mu_B(a) = (\mu_A \cap \mu_B)(a) \end{aligned}$$

and

$$\begin{aligned} (\gamma_A \circ \gamma_B)(a) &= \bigwedge_{a=(ax)a} \{\gamma_A(ax) \vee \gamma_B(a)\} \leq \gamma_A(ax) \vee \gamma_B(a) \\ &\leq \gamma_A(a) \vee \gamma_B(a) = (\gamma_A \cup \gamma_B)(a). \end{aligned}$$

Thus we get that $\mu_A \circ \mu_B \supseteq \mu_A \cap \mu_B$ and $\gamma_A \circ \gamma_B \subseteq \gamma_A \cup \gamma_B$, which give us $A \circ B \supseteq A \cap B$ and therefore $A \circ B = A \cap B$. \square

Example 2. *Let us consider an AG-groupoid $S = \{a, b, c, d, e\}$ with left identity d in the following Cayley's table.*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Note that S is not regular, because $c \in S$ is not regular

The converse of Lemma 2 is not true in general which is discussed in the following.

Let us define an *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S in Example 2 as follows: $\mu_A(a) = \mu_A(b) = \mu_A(c) = 0.3$, $\mu_A(d) = 0.1$, $\mu_A(e) = 0.4$, $\gamma_A(a) = 0.2$, $\gamma_A(b) = 0.3$, $\gamma_A(c) = 0.4$, $\gamma_A(d) = 0.5$, $\gamma_A(e) = 0.2$. Then it is easy to see that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two sided ideals of S . Now again define an *IFS* $B = (\mu_B, \gamma_B)$ of an AG-groupoid S in Example 2 as follows: $\mu_B(a) = \mu_B(b) = \mu_B(c) = 0.5$, $\mu_B(d) = 0.4$, $\mu_B(e) = 0.6$, $\gamma_B(a) = 0.3$, $\gamma_B(b) = 0.4$, $\gamma_B(c) = 0.5$, $\gamma_B(d) = 0.6$, $\gamma_B(e) = 0.3$. Then it is easy to observe that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two sided ideals of S such that $(\mu_A \circ \mu_B)(a) = \{0.1, 0.3, 0.4\} = (\mu_A \cap \mu_B)(a)$ for all $a \in S$ and similarly $(\gamma_A \circ \gamma_B)(a) = (\gamma_A \cap \gamma_B)$ for all $a \in S$, that is, $A \circ B = A \cap B$ but S is not regular.

An *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid is said to be idempotent if $\mu_A \circ \mu_A = \mu_A$ and $\gamma_A \circ \gamma_A = \gamma_A$, that is, $A \circ A = A$ or $A^2 = A$.

Lemma 3. *Every intuitionistic fuzzy two-sided ideal $A = (\mu_A, \gamma_A)$ of a regular AG-groupoid is idempotent.*

Proof. Let S be a regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy two-sided ideal of S . Now for $a \in S$ there exists $x \in S$ such that $a = (ax)a$ and therefore, we have

$$\begin{aligned} (\mu_A \circ \mu_A)(a) &= \bigvee_{a=(ax)a} \{\mu_A(ax) \wedge \mu_A(a)\} \geq \mu_A(ax) \wedge \mu_A(a) \\ &\geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a). \end{aligned}$$

Which shows that $\mu_A \circ \mu_A \supseteq \mu_A$ and by using Lemma 1, $\mu_A \circ \mu_A \subseteq \mu_A$ and therefore $\mu_A \circ \mu_A = \mu_A$. Similarly we can show that $\gamma_A \circ \gamma_A = \gamma_A$, which shows that $A = (\mu_A, \gamma_A)$ is idempotent. \square

Lemma 4. *In a regular AG-groupoid S , $A \circ \delta = A$ and $\delta \circ A = A$ holds for every intuitionistic fuzzy two-sided ideal $A = (\mu_A, \gamma_A)$ of S , where $\delta = (S_\delta, \Theta_\delta)$.*

Proof. Let S be a regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy two-sided ideal of S . Now for $a \in S$ there exists $x \in S$ such that $a = (ax)a$, therefore

$$\begin{aligned} (\mu_A \circ S_\delta)(a) &= \bigvee_{a=(ax)a} \{\mu_A(ax) \wedge S_\delta(a)\} \geq \mu_A(ax) \wedge S_\delta(a) \\ &\geq \mu_A(a) \wedge 1 = \mu_A(a) \end{aligned}$$

and

$$\begin{aligned} (\gamma_A \circ \Theta_\delta)(a) &= \bigwedge_{a=(ax)a} \{\gamma_A(ax) \wedge \Theta_\delta(a)\} \leq \gamma_A(ax) \wedge \Theta_\delta(a) \\ &\leq \gamma_A(a) \wedge 0 = \gamma_A(a). \end{aligned}$$

Which shows that $\mu_A \circ S \supseteq \mu_A$ and $\gamma_A \circ \Theta_\delta \subseteq \gamma_A$. Now by using Lemma 1, we get $\mu_A \circ S = \mu_A$ and $\gamma_A \circ \Theta_\delta = \gamma_A$. Therefore $A \circ \delta = A$. Similarly we can prove that $S \circ A = A$. \square

Corollary 1. *In a regular AG-groupoid S , $A \circ \delta = A$ and $\delta \circ A = A$ hold for every intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of S , where $\delta = (S_\delta, \Theta_\delta)$.*

Theorem 2. *The set of intuitionistic fuzzy two-sided ideals of a regular AG-groupoid S forms a semilattice structure with identity δ , where $\delta = (S_\delta, \Theta_\delta)$.*

Proof. Let $\mathbb{I}_{\mu\gamma}$ be the set of intuitionistic fuzzy two-sided ideals of a regular AG-groupoid S and let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ are any intuitionistic fuzzy two sided ideals of $\mathbb{I}_{\mu\gamma}$. Clearly $\mathbb{I}_{\mu\gamma}$ is closed and by Lemma 3, we have $A = A^2$. Now by using Lemma 2, we get $A \circ B = B \circ A$ and therefore, we have

$$(A \circ B) \circ C = (B \circ A) \circ C = (C \circ A) \circ B = (A \circ C) \circ B = (B \circ C) \circ A = A \circ (B \circ C).$$

It is easy to see from Lemma 4 that δ is an identity in $\mathbb{I}_{\mu\gamma}$. \square

Lemma 5. *Every intuitionistic fuzzy right ideal of an AG-groupoid S with left identity is an intuitionistic fuzzy left ideal of S .*

Proof. Let S be an AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of S . Now by using (1), we have

$$\mu_A(ab) = \mu_A((ea)b) = \mu_A((ba)e) \geq \mu_A(b).$$

Similarly we can show that $\gamma_A(ab) \leq \gamma_A(b)$, which shows that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S . \square

The converse is not true in general because if we define an *IFS* $A = (\mu_A, \gamma_A)$ of an AG-groupoid S in Example 2 as follows: $\mu_A(a) = 0.8, \mu_A(b) = 0.5, \mu_A(c) = 0.4, \mu_A(d) = 0.3, \mu_A(e) = 0.6, \gamma_A(a) = 0.1, \gamma_A(b) = 0.7, \gamma_A(c) = 0.6, \gamma_A(d) = 0.8$ and $\gamma_A(e) = 0.3$, then it is easy to observe that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S but it is not an intuitionistic fuzzy right ideal of S , because $\mu_A(bd) \not\geq \mu_A(b)$ and $\gamma_A(cd) \not\leq \gamma_A(c)$.

Corollary 2. *Every intuitionistic fuzzy right ideal of a regular AG-groupoid S with left identity is an intuitionistic fuzzy left ideal of S .*

To consider the converse of Corollary 2, we need to strengthen the condition of a regular AG-groupoid S which is given in the following.

An AG-groupoid S is called a left duo if every left ideal of S is a two-sided ideal of S .

Lemma 6. *Let S be a regular AG-groupoid such that S is a left duo, then every intuitionistic fuzzy left ideal of S is an intuitionistic fuzzy right ideal of S .*

Proof. Let S be a left duo regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S . Let $x, y \in S$ then the left ideal Sx of S is a two sided ideal of S and since S is regular therefore by using (1), we have

$$xy \in ((xS)x)y \subseteq ((xS)((xS)x))S \subseteq (((xS)x)S)xS \subseteq (Sx)S \subseteq S.$$

It follows that there exists $w \in S$ such that $xy = wx$. As $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S , therefore we get $\mu_A(xy) = \mu_A(wx) \geq \mu_A(x)$ and $\mu_A(xy) = \gamma_A(wx) \leq \gamma_A(x)$. This means that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S . \square

An AG-groupoid S is called an intuitionistic fuzzy left duo if every intuitionistic fuzzy left ideal of S is an intuitionistic fuzzy two-sided ideal of S .

Corollary 3. *Let S be a regular AG-groupoid. If S is a left duo, then S is an intuitionistic fuzzy left duo.*

Theorem 3. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of a regular AG-groupoid S with left identity, then $A(ab) = A(ba)$ holds for all a, b in S .*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy two-sided ideal of a regular AG-groupoid S with left identity and let $a, b \in S$, then $a = (ax)a$ and $b = (by)b$ for some $x, y \in S$. Now by using (2) and (3), we have

$$\begin{aligned} \mu_A(ab) &= \mu_A(((ax)a)((by)b)) = \mu_A(((ax)(by))(ab)) = \mu_A((ba)((by)(ax))) \\ &\geq \mu_A(ba) = \mu_A(((by)b)((ax)a)) = \mu_A(((by)(ax))(ba)) \\ &= \mu_A((ab)((ax)(by))) \geq \mu_A(ab). \end{aligned}$$

Which shows that $\mu_A(ab) = \mu_A(ba)$ holds for all a, b in S and similarly $\gamma_A(ab) = \gamma_A(ba)$ holds for all a, b in S . Thus $A(ab) = A(ba)$ holds for all a, b in S . \square

The converse is not true in general. For this consider an *IFS* $A = (\mu_A, \gamma_A)$ of a regular AG-groupoid S considered in Example 1 as follows: $\mu_A(a) = 0.1$, $\mu_A(b) = 0.2$, $\mu_A(c) = 0.6$, $\mu_A(d) = 0.4$, $\mu_A(e) = 0.6$, $\gamma_A(a) = 0.2$, $\gamma_A(b) = 0.3$, $\gamma_A(c) = 0.7$, $\gamma_A(d) = 0.5$, $\gamma_A(e) = 0.7$, then it is easy to observe that $A(ab) = A(ba)$ holds for all a and b in S but $A = (\mu_A, \gamma_A)$ is not an intuitionistic fuzzy two-sided ideal of S , because $\mu_A(cc) \not\geq \mu_A(c)$ and $\gamma_A(ed) \not\leq \gamma(d)$ ($\gamma_A(de) \not\leq \gamma(d)$).

Corollary 4. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of a regular AG-groupoid S with left identity, then $A(ab) = A(ba)$ holds for all a, b in S .*

The converse of Corollary 4 is not true in general which can be followed from the converse of Theorem 3.

Theorem 4. *Let S be a regular AG-groupoid with left identity, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S .*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of a regular AG-groupoid S and let $w, x, y \in S$, then by using (1), we have

$$\mu_A((xw)y) = \mu_A(((yw)x)) \geq \mu_A(x) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A((xw)y) = \gamma_A(((yw)x)) \leq \mu_A(x) \leq \mu_A(x) \vee \mu_A(y).$$

Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S .

Conversely, let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy bi-ideal of S and let $x, y \in S$, then there exists $z \in S$ such that $y = (yz)y$. Now by using (4), (2), (1) and (3), we have

$$\begin{aligned} \mu_A(xy) &= \mu_A(x((yz)y)) = \mu_A((yz)(xy)) = \mu_A((yx)(zy)) = \mu_A(((zy)x)y) \\ &= \mu_A(((zy)(ex))y) = \mu_A(((xe)(yz))y) = \mu_A((y((xe)z))y) \geq \mu_A(y). \end{aligned}$$

Similarly $\gamma_A(xy) \leq \gamma_A(y)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S . \square

Corollary 5. *Let S be a regular AG-groupoid with left identity, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of S .*

Theorem 5. *Let S be a regular AG-groupoid with left identity such that S is a left duo, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S .*

Proof. It follows from Theorem 4 and Lemma 6. \square

Corollary 6. *Let S be a regular AG-groupoid with left identity such that S is a left duo, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of S .*

Theorem 6. *Let S be a regular AG-groupoid with left identity, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1, 2)-ideal of S if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S .*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of a regular AG-groupoid S and let $w, x, y, z \in S$, then there exists $a, b \in S$ such that $x = (xa)x$ and $y = (yb)y$. Now by using (1) and (4), we have

$$\begin{aligned}\mu_A((xw)(yz)) &= \mu_A((((xa)x)w)(yz)) = \mu_A(((wx)(xa))(yz)) \\ &= \mu_A((x((wx)a))(yz)) = \mu_A(((yz)((wx)a))x) \geq \mu_A(x).\end{aligned}$$

Now by using (4), (2) and (1), we have

$$\begin{aligned}\mu_A((xw)(yz)) &= \mu_A(y((wx)z)) = \mu_A(((yb)y)((wx)z)) = \mu_A(((yb)(wx))(yz)) \\ &= \mu_A((((yb)y)b)(wx))(yz)) = \mu_A((((by)(yb))(wx))(yz)) \\ &= \mu_A(((y((by)b))(wx))(yz)) = \mu_A(((wx)((by)b))y)(yz)) \\ &= \mu_A((((by)((wx)b))y)(yz)) = \mu_A(((y((wx)b))(yb))(yz)) \\ &= \mu_A(((y((y((wx)b)b))(yz)) = \mu_A(((yz)(y((wx)b)b))y) \geq \mu_A(y).\end{aligned}$$

Now by using (3) and (1), we have

$$\mu_A((xw)(yz)) = \mu_A((zy)(wx)) = \mu_A(((wx)y)z) \geq \mu_A(z).$$

Thus we get, $\mu_A((xw)(yz)) \geq \mu_A(x) \wedge \mu_A(y) \wedge \mu_A(z)$ and similarly $\gamma_A((xw)(yz)) \leq \gamma_A(x) \vee \gamma_A(y) \vee \gamma_A(z)$. Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1,2)-ideal of S . \square

For the converse of Theorem 6, we have to strengthen the condition of a regular AG-groupoid which is given in the following.

An AG-groupoid S is called an AG-band if $a = a^2$ for all $a \in S$.

Theorem 7. *Let S be a regular AG-band, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1,2)-ideal of S .*

Proof. It is simple. \square

Theorem 8. *Let S be a regular AG-groupoid with left identity such that S is a left duo, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1,2)-ideal of S if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S .*

Proof. It is an easy consequence of Theorem 6 and Lemma 6. \square

Theorem 9. *Let S be a regular AG-band, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1,2)-ideal of S .*

Proof. It is simple. \square

Lemma 7. ([13],[9]) *For any IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S , the following properties holds.*

(i) A is an AG-subgroupoid of S if and only if χ_A is an intuitionistic fuzzy AG-subgroupoid of S .

(ii) A is an intuitionistic left (right, two-sided) ideal of S if and only if χ_A is an intuitionistic fuzzy left (right, two-sided) ideal of S .

A subset A of an AG-groupoid S is called semiprime if $a^2 \in A$ implies $a \in A$. An IFS $A = (\mu_A, \gamma_A)$ of an AG-groupoid S is called an intuitionistic fuzzy semiprime if $\mu_A(a) \geq \mu_A(a^2)$ and $\gamma_A(a) \leq \gamma_A(a^2)$ for all a in S .

Lemma 8. *Every right (left, two-sided ideal) of an AG-groupoid S is semiprime if and only if their characteristic functions are intuitionistic fuzzy semiprime.*

Proof. Let R be any right ideal of an AG-groupoid S , then by Lemma 7, the intuitionistic characteristic function of R , that is, $\chi_R = (\mu_{\chi_R}, \gamma_{\chi_R})$ is an intuitionistic fuzzy right ideal of S . Let $a^2 \in R$, then $\mu_{\chi_R}(a^2) = 1$ and assume that R is semiprime, then $a \in R$, which implies that $\mu_{\chi_R}(a) = 1$. Thus we get $\mu_{\chi_R}(a^2) = \mu_{\chi_R}(a)$ and similarly we can show that $\gamma_{\chi_R}(a^2) = \gamma_{\chi_R}(a)$, therefore $\chi_R = (\mu_{\chi_R}, \gamma_{\chi_R})$ is an intuitionistic fuzzy semiprime. The converse is simple. \square

Corollary 7. *Let S be an AG-groupoid, then every right (left, two-sided) ideal of S is semiprime if every intuitionistic fuzzy right (left, two-sided) ideal of S is an intuitionistic fuzzy semiprime.*

The converse is not true in general. For this let us consider an AG-groupoid S in Example 2. It is easy to observe that the only left ideals of S are $\{a, b, e\}$, $\{a, c, e\}$, $\{a, b, c, e\}$ and $\{a, e\}$ which are semiprime. Clearly the right and two sided ideals of S are $\{a, b, c, e\}$ and $\{a, e\}$ which are also semiprime. Now on the other hand, if we define an *IFS* $A = (\mu_A, \gamma_A)$ of S as follows: $\mu_A(a) = \mu_A(b) = \mu_A(c) = 0.2$, $\mu_A(d) = 0.1$, $\mu_A(e) = 0.3$, $\gamma_A(a) = 0.2$, $\gamma_A(b) = \gamma_A(c) = 0.5$, $\gamma_A(d) = 0.6$ and $\gamma_A(e) = 0.3$, then $A = (\mu_A, \gamma_A)$ is a fuzzy right (left, two-sided) ideal of S but $A = (\mu_A, \gamma_A)$ is not an intuitionistic fuzzy semiprime, because $\mu_A(c) \not\geq \mu_A(c^2)$ and $\gamma_A(c) \not\leq \gamma_A(c^2)$.

An element a of an AG-groupoid S is called an intra-regular if there exist $x, y \in S$ such that $a = (xa^2)y$ and S is called an intra-regular if every element of S is an intra-regular.

Example 3. *Let $S = \{a, b, c, d, e\}$ be an AG-groupoid with left identity b in the following cayley's table.*

.	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

Clearly S is an intra-regular because, $a = (aa^2)a$, $b = (cb^2)e$, $c = (dc^2)e$, $d = (cd^2)c$ and $e = (be^2)e$.

Lemma 9. *For an intra-regular AG-groupoid S with left identity, the following holds.*

- (i) Every intuitionistic fuzzy right ideal of S is an intuitionistic fuzzy semiprime.
- (ii) Every intuitionistic fuzzy left ideal of S is an intuitionistic fuzzy semiprime.
- (iii) Every intuitionistic fuzzy two-sided ideal of S is an intuitionistic fuzzy semiprime.

Proof. (i) : Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of an intra-regular AG-groupoid S with left identity and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (3) and (4), we have

$$\mu_A(a) = \mu_A((xa^2)y) = \mu_A((xa^2)(ey)) = \mu_A((ye)(a^2x)) = \mu_A(a^2((ye)x)) \geq \mu_A(a^2)$$

and similarly

$$\gamma_A(a) = \gamma_A((xa^2)y) = \gamma_A((xa^2)(ey)) = \gamma_A((ye)(a^2x)) = \gamma_A(a^2((ye)x)) \leq \gamma_A(a^2).$$

Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy semiprime.

(ii) : Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of an intra-regular AG-groupoid S with left identity and let $a \in S$, then there exist $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (3) and (1), we have

$$\begin{aligned}\mu_A(a) &= \mu_A((xa^2)y) = \mu_A((x(aa))y) = \mu_A((a(xa))y) = \mu_A((((xa^2)y)(xa))y) \\ &= \mu_A(((ax)(y(xa^2)))y) = \mu_A(((ax)(y((ex)(aa))))y) \\ &= \mu_A(((ax)(y(a^2(xe))))y) = \mu_A(((ax)(a^2(y(xe))))y) \\ &= \mu_A(a^2((ax)(y(xe)))y) = \mu_A((y((y(xe))(ax)))a^2) \geq \mu_A(a^2).\end{aligned}$$

Similarly we can show that $\gamma_A(a) \leq \gamma_A(a^2)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy semiprime.

(iii) : It can be followed from (i) and (ii). \square

Theorem 10. *Let S be an AG-groupoid with left identity, then the following statements are equivalent.*

(i) S is intra-regular.

(ii) Every intuitionistic fuzzy right (left, two-sided) ideal of S is an intuitionistic fuzzy semiprime.

Proof. (i) \implies (ii) is followed by Lemma 9.

(ii) \implies (i) : Let S be an AG-groupoid with left identity and let every intuitionistic fuzzy right (left, two-sided) ideal of S is an intuitionistic fuzzy semiprime. Since a^2S is a right and also a left ideal of S , therefore by using Corollary 7, a^2S is semiprime. Clearly $a^2 \in a^2S$ and therefore $a \in a^2S$. Now by using (1), we have

$$a \in a^2S = (aa)S = (Sa)a \subseteq (Sa)(a^2S) = ((a^2S)a)S = ((aS)a^2)S \subseteq (Sa^2)S.$$

Which shows that S is an intra-regular. \square

Theorem 11. *The following statements are equivalent for an AG-groupoid with left identity.*

(i) S is an intra-regular.

(ii) Every intuitionistic fuzzy right ideal of S is an intuitionistic fuzzy semiprime.

(iii) Every intuitionistic fuzzy left ideal of S is an intuitionistic fuzzy semiprime.

Proof. (i) \implies (iii) and (ii) \implies (i) are followed by Theorem 10.

(iii) \implies (ii) : Let S be an AG-groupoid with left identity and let every intuitionistic fuzzy left ideal of S is an intuitionistic fuzzy semiprime. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of S . Now by using Lemma 5, $A = (\mu_A, \gamma_A)$ is an intuitionistic \square

Lemma 10. ([13],[9]) *For any non-empty IFSs $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ of an AG-groupoid S , then $\chi_A \circ \chi_B = \chi_{AB}$.*

Theorem 12. *For an AG-groupoid S with left identity, the following conditions are equivalent.*

(i) S is an intra-regular.

(ii) $R \cap L = RL$, R is any right ideal and L is any left ideal of S such that R is semiprime.

(iii) $A \cap B = A \circ B$, $A = (\mu_A, \gamma_A)$ is any intuitionistic fuzzy right ideal and $B = (\mu_B, \gamma_B)$ is any intuitionistic fuzzy left ideal of S such that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy semiprime.

Proof. (i) \implies (iii) : Assume that S is an intra-regular AG-groupoid. Let $A = (\mu_A, \gamma_A)$ is any intuitionistic fuzzy right ideal and $B = (\mu_B, \gamma_B)$ is any intuitionistic fuzzy left ideal of S . Now for $a \in S$ there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (1) and (3), we have

$$\begin{aligned} a &= (x(aa))y = (a(xa))y = (y(xa))a = (y(x((xa^2)y)))a = (y((xa^2)(xy)))a \\ &= (y((yx)(a^2x)))a = (y(a^2((yx)x)))a = (a^2(y((yx)x)))a. \end{aligned}$$

Therefore

$$\begin{aligned} (\mu_A \circ \mu_B)(a) &= \bigvee_{a=(a^2(y((yx)x)))a} \{\mu_A(a^2(y((yx)x))) \wedge \mu_B(a)\} \\ &\geq \mu_A(a) \wedge \mu_B(a) = (\mu_A \cap \mu_B)(a) \end{aligned}$$

and

$$\begin{aligned} (\gamma_A \circ \gamma_B)(a) &= \bigwedge_{a=(a^2(y((yx)x)))a} \{\gamma_A(a^2(y((yx)x))) \vee \gamma_B(a)\} \\ &\leq \gamma_A(a) \wedge \gamma_B(a) = (\gamma_A \cup \gamma_B)(a). \end{aligned}$$

Which implies that $A \circ B \supseteq A \cap B$ and by using Lemma 1, $A \circ B \subseteq A \cap B$, therefore $A \cap B = A \circ B$.

(iii) \implies (ii)

Let R be any right ideal and L be any left ideal of an AG-groupoid S , then by Lemma 7, $\chi_R = (\mu_{\chi_R}, \gamma_{\chi_R})$ and $\chi_L = (\mu_{\chi_L}, \gamma_{\chi_L})$ are an intuitionistic fuzzy right and intuitionistic fuzzy left ideals of S respectively. As $RL \subseteq R \cap L$ is obvious therefore let $a \in R \cap L$, then $a \in R$ and $a \in L$. Now by using Lemma 10 and given assumption, we have

$$\mu_{\chi_{RL}}(a) = (\mu_{\chi_R} \circ \mu_{\chi_L})(a) = (\mu_{\chi_R} \cap \mu_{\chi_L})(a) = \mu_{\chi_R}(a) \wedge \mu_{\chi_L}(a) = 1$$

and similarly

$$\gamma_{\chi_{RL}}(a) = (\gamma_{\chi_R} \circ \gamma_{\chi_L})(a) = (\gamma_{\chi_R} \cup \gamma_{\chi_L})(a) = \gamma_{\chi_R}(a) \vee \gamma_{\chi_L}(a) = 1.$$

Which implies that $a \in RL$ and therefore $R \cap L = RL$. Now by using Corollary 7, R is semiprime.

(ii) \implies (i)

Let S be an AG-groupoid, then clearly Sa is a left ideal of S such that $a \in Sa$ and a^2S is a right ideal of S such that $a^2 \in a^2S$. Since by assumption, a^2S is semiprime, therefore $a \in a^2S$. Now by using (3), (1) and (4), we have

$$\begin{aligned} a &\in a^2S \cap Sa = (a^2S)(Sa) = (aS)(Sa^2) = ((Sa^2)S)a = ((Sa^2)(SS))a \\ &= ((SS)(a^2S))a = (a^2((SS)S))a \subseteq (a^2S)S = (SS)(aa) = a^2S \\ &= (aa)S = (Sa)a \subseteq (Sa)(a^2S) = ((a^2S)a)S = ((aS)a^2)S \subseteq (Sa^2)S. \end{aligned}$$

Which shows that S is an intra-regular. \square

Lemma 11. Let $A = (\mu_A, \gamma_A)$ be an IFS of an intra-regular AG-groupoid S with left identity, then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S if and only if $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S .

Proof. Let S be an intra-regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S . Now for $a, b \in S$ there exists $x, y, x', y' \in S$ such that $a = (xa^2)y$ and $b = (x'b^2)y'$, then by using (1), (3) and (4), we have

$$\begin{aligned}\mu_A(ab) &= \mu_A(((xa^2)y)b) = \mu_A((by)(x(aa))) = \mu_A(((aa)x)(yb)) \\ &= \mu_A(((xa)a)(yb)) = \mu_A(((xa)(ea))(yb)) = \mu_A(((ae)(ax))(yb)) \\ &= \mu_A((a((ae)x))(yb)) = \mu_A(((yb)((ae)x))a) \geq \mu_A(a).\end{aligned}$$

Similarly we can get $\gamma_A(ab) \leq \gamma_A(a)$, which implies that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S .

Conversely let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of S . Now by using (4) and (3), we have

$$\begin{aligned}\mu_A(ab) &= \mu_A(a((x'b^2)y')) = \mu_A((x'b^2)(ay')) = \mu_A((y'a)(b^2x')) \\ &= \mu_A(b^2((y'a)x)) \geq \mu_A(b).\end{aligned}$$

Similarly we can get $\gamma_A(ab) \leq \gamma_A(b)$, which implies that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S . \square

Theorem 13. *Let S be an intra-regular AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ be an IFS, then the following conditions are equivalent.*

- (i) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S .
- (ii) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S .

Proof. (i) \implies (ii) is simple.

(ii) \implies (i) : Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of an intra-regular AG-groupoid S and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (1) and (3), we have

$$\begin{aligned}\mu_A(ab) &= \mu_A(((x(aa))y)b) = \mu_A(((a(xa))y)b) = \mu_A((by)((ea)(xa))) \\ &= \mu_A((by)((ax)(ae))) = \mu_A(((ae)(ax))(yb)) = \mu_A((a((ae)x))(yb)) \\ &= \mu_A(((yb)((ae)x))a) = \mu_A(((yb)((x(a^2)y)e)x)a) \\ &= \mu_A(((yb)((y(xa^2))(ex))a) = \mu_A(((yb)((xe)((a^2)(ey))))a) \\ &= \mu_A(((yb)((xe)((ye)(a^2x))))a) = \mu_A(((yb)((xe)(a^2((ye)x))))a) \\ &= \mu_A(((yb)(a^2((xe)((ye)x))))a) = \mu_A((a^2((yb)((xe)((ye)x))))a) \\ &\geq \mu_A(a^2) \wedge \mu_A(a) \geq \mu_A(a) \wedge \mu_A(a) \wedge \mu_A(a) = \mu_A(a).\end{aligned}$$

Similarly we can show that $\gamma_A(ab) \leq \gamma_A(a)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S . Now by using Lemma 11, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S . \square

Theorem 14. *Let S be an intra-regular AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ be an IFS, then the following conditions are equivalent.*

- (i) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S .
- (ii) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (1,2)-ideal of S .

Proof. (i) \implies (ii) is simple.

(ii) \implies (i) : Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of an intra-regular AG-groupoid S and let $b \in S$, then there exists $x, y \in S$ such that

$b = (xb^2)y$. Now by using (4), (1) and (3), we have

$$\begin{aligned}
\mu_A(ab) &= \mu_A(a((xb^2)y)) = \mu_A((x(bb))(ay)) = \mu_A((b(xb))(ay)) \\
&= \mu_A(((ay)(xb))b) = \mu_A(((ay)(xb))((xb^2)y)) = \mu_A((xb^2)(((ay)(xb))y)) \\
&= \mu_A((y((ay)(xb)))(b^2x)) = \mu_A(b^2((y((ay)(xb)))x)) \\
&= \mu_A((bb)((y((ay)(xb)))x)) = \mu_A((x(y((ay)(xb))))(bb)) \\
&= \mu_A((x(y((bx)(ya))))(bb)) = \mu_A((x((bx)(y(ya))))(bb)) \\
&= \mu_A(((bx)(x(y(ya))))(bb)) = \mu_A((((xb^2)y)x)(x(y(ya))))(bb)) \\
&= \mu_A(((xy)(xb^2))(x(y(ya))))(bb)) = \mu_A((((b^2x)(yx))(x(y(ya))))(bb)) \\
&= \mu_A((((yx)x)b^2)(x(y(ya))))(bb)) = \mu_A((((y(ya)x)(b^2((yx)x))))(bb)) \\
&= \mu_A((((y(ya)x)(b^2(x^2y))))(bb)) = \mu_A((b^2(((y(ya)x)(x^2y))))(bb)) \\
&= \mu_A((((x^2y)((y(ya)x)))(bb))(bb)) = \mu_A((a((x^2y)((y(ya)x)b)))(bb)) \\
&\geq \mu_A(b) \wedge \mu_A(b) \wedge \mu_A(b) = \mu_A(b).
\end{aligned}$$

Similarly we can show that $\gamma_A(ab) \leq \gamma_A(b)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal of S . Now by using Lemma 11, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy two-sided ideal of S . \square

Theorem 15. *Let S be an intra-regular AG-groupoid with left identity and let $A = (\mu_A, \gamma_A)$ be an IFS, then the following conditions are equivalent.*

- (i) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S .
- (ii) $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy generalized bi-ideal of S .

Proof. (i) \implies (ii) is obvious.

(ii) \implies (i) : Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy generalized bi-ideal of an intra-regular AG-groupoid S and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (3) and (1), we have

$$\begin{aligned}
\mu_A(ab) &= \mu_A(((x(aa))y)b) = \mu_A((((ea)(xa))y)b) = \mu_A((((ax)(ae))y)b) \\
&= \mu_A(((a((ax)e))(ey))b) = \mu_A((((ye)((ax)e)a))b) \\
&= \mu_A((((ye)((ae)(ax)))b) = \mu_A((((ye)(a((ae)x)))b) \\
&= \mu_A((a((ye)((ae)x)))b) \geq \mu_A(a) \wedge \mu_A(b).
\end{aligned}$$

Similarly we can show that $\gamma_A(ab) \leq \gamma_A(a) \vee \gamma_A(b)$ and therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy bi-ideal of S . \square

Theorem 16. *An AG-groupoid S is an intra-regular if and only if for each intuitionistic fuzzy right (left, two-sided) ideal $A = (\mu_A, \gamma_A)$ of S , $A(a) = A(a^2)$ for all a in S .*

Proof. Assume that S be an intra-regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy right ideal of S . Let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$.

$$\begin{aligned}
\mu_A(a) &= \mu_A((xa^2)y) = \mu_A((xa^2)(ey)) = \mu_A((ye)(a^2x)) = \mu_A(a^2((ye)x)) \\
&\geq \mu_A(a^2) = \mu_A(aa) \geq \mu_A(a).
\end{aligned}$$

Similarly we can show that $\gamma_A(a) = \gamma_A(a)$ and therefore $A(a) = A(a^2)$ holds for all a in S .

Conversely, assume that for any intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of S , $A(a) = A(a^2)$ holds for all a in S . As a^2S is a right and also a left ideal of S , then by Lemma 7, $\chi_{a^2S} = (\mu_{\chi_{a^2S}}, \gamma_{\chi_{a^2S}})$ is an intuitionistic fuzzy right and an intuitionistic fuzzy left ideal of S and therefore by given assumption and using the fact that $a^2 \in a^2S$, we have $\mu_{\chi_{a^2S}}(a) = \mu_{\chi_{a^2S}}(a^2) = 1$ and $\gamma_{\chi_{a^2S}}(a) = \gamma_{\chi_{a^2S}}(a^2) = 0$, which implies that $a \in a^2S$. Now by using (4) and (2), we have $a \in (Sa^2)S$ and therefore S is an intra-regular. \square

Theorem 17. *Let S be an AG-groupoid with left identity, then the following conditions are equivalent.*

- (i) S is an intra-regular.
- (ii) Every intuitionistic fuzzy left (right, two-sided) ideal of S is idempotent.

Proof. (i) \implies (ii) : Let S be an intra-regular AG-groupoid with left identity and let $a \in S$, then there exists $x, y \in S$ such that $a = (xa^2)y$. Now by using (4), (1) and (3), we have

$$a = (x(aa))y = (a(xa))y = (y(xa))a = ((ex)(ya))a = ((ay)(xe))a = (((xe)y)a)a.$$

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal of S , then by using Lemma 7, we have $\mu_A \circ \mu_A \subseteq \mu_A$ and also we have

$$(\mu_A \circ \mu_A)(a) = \bigvee_{a=(((xe)y)a)a} \{\mu_A(((xe)y)a) \wedge \mu_A(a)\} \geq \mu_A(a) \wedge \mu_A(a) = \mu_A(a).$$

Which implies that $\mu_A \circ \mu_A \supseteq \mu_A$ and similarly we can get $\gamma_A \circ \gamma_A \subseteq \gamma_A$. Now by using Lemma 7, $\mu_A \circ \mu_A \subseteq \mu_A$ and $\gamma_A \circ \gamma_A \supseteq \gamma_A$. Thus that $A = (\mu_A, \gamma_A)$ is idempotent and by using Lemma 11, every intuitionistic fuzzy right and two-sided is idempotent.

(ii) \implies (i)

Assume that every left ideal of an AG-groupoid S with left identity is idempotent and let $a \in S$. Since Sa is a left ideal of S , therefore by Lemma 7, its characteristic function $\chi_{Sa} = (\mu_{\chi_{Sa}}, \gamma_{\chi_{Sa}})$ is an intuitionistic fuzzy left ideal of S . Since $a \in Sa$ therefore $\mu_{\chi_{Sa}}(a) = 1$. and $\gamma_{\chi_{Sa}}(a) = 0$. Now by using the given assumption and Lemma 10, we have

$$\mu_{\chi_{Sa}} \circ \mu_{\chi_{Sa}} = \mu_{\chi_{Sa}} \text{ and } \mu_{\chi_{Sa}} \circ \mu_{\chi_{Sa}} = \mu_{\chi_{(Sa)^2}}.$$

Thus we have $(\mu_{\chi_{(Sa)^2}})(a) = (\mu_{\chi_{Sa}})(a) = 1$ and similarly we can get $(\gamma_{\chi_{(Sa)^2}})(a) = (\gamma_{\chi_{Sa}})(a) = 0$, which implies that $a \in (Sa)^2$. Now by using (1), (2) and (3), we have

$$\begin{aligned} a &\in (Sa)^2 = (Sa)(Sa) = ((Sa)a)S \subseteq ((Sa)((Sa)(Sa)))S = ((Sa)((SS)(aa)))S \\ &= ((Sa)(Sa^2))S = ((a^2S)(aS))S = (((aS)S)a^2)S \subseteq (Sa^2)S. \end{aligned}$$

Which shows that S is an intra-regular.

Let every right ideal of an AG-groupoid S with left identity is idempotent $a \in S$. Clearly a^2S is a right ideal of S , therefore by Lemma 7, $\chi_{a^2S} = (\mu_{\chi_{a^2S}}, \gamma_{\chi_{a^2S}})$ is an intuitionistic fuzzy right ideal of S . As $a \in a^2S$ therefore $\mu_{\chi_{a^2S}}(a) = 1$ and $\gamma_{\chi_{a^2S}}(a) = 0$. Now by using the given assumption and Lemma 10, we have

$$\mu_{\chi_{a^2S}} \circ \mu_{\chi_{a^2S}} = \mu_{\chi_{a^2S}} \text{ and } \mu_{\chi_{a^2S}} \circ \mu_{\chi_{a^2S}} = \mu_{\chi_{a^2S}}.$$

Thus we get $(\mu_{\chi_{a^2S}})(a) = (\mu_{\chi_{a^2S}})(a) = 1$ and similarly we can get $(\gamma_{\chi_{a^2S}})(a) = (\gamma_{\chi_{a^2S}})(a) = 0$, which implies that $a \in (a^2S)^2$. Now by using (3) and (1), we have

$$a \in (a^2S)^2 = (a^2S)(a^2S) = (Sa^2)(Sa^2) = ((Sa^2)a^2)S \subseteq (Sa^2)S.$$

Which shows that S is an intra-regular. \square

Note that if an AG-groupoid S contains a left identity, then $S = S \circ S$.

Theorem 18. *For an AG-groupoid S with left identity, the following conditions are equivalent.*

- (i) S is an intra-regular.
- (ii) $A = (S \circ A)^2$, where $A = (\mu_A, \gamma_A)$ is any intuitionistic fuzzy left (right, two-sided) ideal of S .

Proof. (i) \implies (ii)

Let S be an intra-regular AG-groupoid and let $A = (\mu_A, \gamma_A)$ be any intuitionistic fuzzy left ideal of S , then it is easy to see that $S \circ A$ is also an intuitionistic fuzzy left ideal of S . Now by using Theorem 17, $S \circ A$ is idempotent and therefore, we have

$$(S \circ A)^2 = S \circ A \subseteq A.$$

Now let $a \in S$, since S is an intra-regular therefore there exists $x \in S$ such that $a = (xa^2)y$ and by using (4), (3) and (1), we have

$$\begin{aligned} a &= (x(aa))y = (a(xa))y = (((xa^2)y)(xa))(ey) = (ye)((xa)((xa^2)y)) \\ &= (xa)((ye)((xa^2)y)) = (xa)((ye)(x(aa)))y = (xa)((ye)(a(xa)))y \\ &= (xa)((a((ye)(xa)))y) = (xa)((y((ye)(xa)))a) = (xa)p \end{aligned}$$

where $p = ((y((ye)(xa)))a)$ and therefore, we have

$$\begin{aligned} (S \circ \mu_A)^2(a) &= \bigvee_{a=(xa)((y((ye)(xa)))a)} \{(S \circ \mu_A)(xa) \wedge (S \circ \mu_A)((y((ye)(xa)))a)\} \\ &\geq (S \circ \mu_A)(xa) \wedge (S \circ \mu_A)((y((ye)(xa)))a) \\ &= \bigvee_{xa=xa} \{S(x) \wedge \mu_A(a)\} \wedge \bigvee_{p=(y((y(xa)))a)} \{S(y((ye)(xa))) \wedge \mu_A(a)\} \\ &\geq S(x) \wedge \mu_A(a) \wedge S(y((ye)(xa))) \wedge \mu_A(a) = \mu_A(a). \end{aligned}$$

Similarly we can get $(S \circ \gamma_A)^2(a) \leq \gamma_A(a)$, which implies that $(S \circ A)^2 \supseteq A$. Thus we get the required $A = (S \circ A)^2$.

(ii) \implies (i)

Let $A = (S \circ A)^2$ holds for any intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ of S , then by using Lemma 1 and given assumption, we have

$$\mu_A = (S \circ \mu_A)^2 \subseteq \mu_A^2 = \mu_A \circ \mu_A \subseteq S \circ \mu_A \subseteq \mu_A.$$

Which shows that $\mu_A = \mu_A \circ \mu_A$, similarly $\gamma_A = \gamma_A \circ \gamma_A$ and therefore $A = A \circ A$. Thus by using Lemma 17, S is an intra-regular.

Let $A = (S \circ A)^2$ holds for any intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ of S , then by using Lemma 1, given assumption and (1), we have

$$\mu_A = (S \circ \mu_A)^2 = ((S \circ S) \circ \mu_A)^2 = ((\mu_A \circ S) \circ S)^2 \subseteq \mu_A^2 = \mu_A \circ \mu_A \subseteq \mu_A \circ S \subseteq \mu_A.$$

Which shows that $\mu_A = \mu_A \circ \mu_A$, similarly $\gamma_A = \gamma_A \circ \gamma_A$ and therefore $A = A \circ A$. Thus by using Lemma 17, S is an intra-regular. \square

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